JOURNAL OF COMPUTATIONAL PHYSICS 37, 259–263 (1980)

Note

Expressions for the Behavior of a Fourier Transform near Its Singularities*

1. *Introduction*

Let $F(y)$ denote the integral

$$F(y) = \int_0^\infty e^{-iuy} f(u)\, du, \tag{1}$$

where y is a real variable. Here we shall be concerned with the behavior of $F(y)$ near its singularities, i.e., the values of y at which $F(y)$ or some of its derivatives are discontinuous. The general nature of our results has been known for some years (Widder [1], Doetsch [2]). The aim of the present note is to put this material in a form that is of help in the calculation of integrals of type (1).

We shall consider only a simple but frequently occurring kind of singularity, namely, the kind that appears in $F(y)$ when the asymptotic expansion of $f(u)$, as $u \to \infty$, is the sum of components of the form $Au^{-\nu} \exp[iuy_1]$. More general results are available (see Handelsman and Lew [3, 4], Bleistein *et al.* [5]).

2. *Behavior of $F(y)$ When the Asymptotic Expansion of $f(u)$ Consists of a Single Series*

In this case we have the following Abelian-type theorem.

THEOREM. *Let $f(u)$ in the integral* (1) *have the asymptotic expansion*

$$f(u) \sim \exp[iuy_1] \sum_{m=1}^{\infty} A_m u^{-\nu_m}, \qquad u \to \infty, \tag{2}$$

where $0 < \nu_1 < \nu_2 < \cdots \nu_m < \cdots$ *and* $\nu_m \to \infty$ *as* $m \to \infty$. *Then the function* $F(y)$ *defined by* (1) *has a singularity at* $y = y_1$. *For any positive integer* M, $F(y)$ *can be written as*

$$F(y) = \sum_{m=1}^{M} \phi_m(y - y_1) + \psi_M(y), \tag{3}$$

* This research was sponsored by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant AFOSR 74-2689.

0021-9991/80/110259–05$02.00/0

where

(i) $\psi_M(y)$ *and its first* N *derivatives are continuous for all real values of* y, N *being defined by*

$$N+1 < v_{M+1} \leqslant N+2. \tag{4}$$

(ii) *the functions* $\phi_m(y-y_1)$ *are defined by*

$$\phi_m(y-y_1) = A_m[i(y-y_1)]^{v_m-1}\pi/[\Gamma(v_m)\sin \pi v_m] \tag{5}$$

when v_m *is not an integer, and by*

$$\phi_m(y-y_1) = A_m(-)^n[i(y-y_1)]^{n-1}\ln[i(y-y_1)]/(n-1)! \tag{6}$$

when v_m *is equal to the integer* $n \geqslant 1$. *In* (5) *and* (6), $\arg[i(y-y_1)]$ *is equal to* $\pi/2$ *when* $y > y_1$ *and to* $-\pi/2$ *when* $y < y_1$.

The theorem can be proved by (i) writing $F(y)$ as the sum of an integral with limits (0, 1) and one with limits $(1, \infty)$, (ii) adding and subtracting N terms of (2) to the integrand of the second integral, then (iii) using properties of the incomplete gamma function and exponential integral.

3. *Behavior of* $F(y)$ *When the Asymptotic Expansion of* $f(u)$ *Consists of the Sum of Several Series*

In this case $F(y)$ is irregular at several values of y corresponding to y_1 (these singularities will be referred to as the "y_1's"). Corresponding to each y_1 there is a set of $\phi_m(y-y_1)$'s. In the neighborhood of a particular y_1, $F(y)$ can still be expressed in form (3) but now, although $\psi_m(y)$ and its first N derivatives are continuous at the particular y_1, they may not be continuous at the remaining y_1,s.

Thus, in general, corresponding to a term $Au^{-v}\exp[iuy_1]$ in the asymptotic expansion of $f(u)$ there is an irregular portion of $F(y)$, namely, the function $\phi(y-y_1)$ given by

$$A\pi[i(y-y_1)]^{v-1}/[\Gamma(v)\sin \pi v], \qquad v>0 \text{ but} \neq 1, 2, 3,..., \tag{7}$$

$$A(-)^n[i(y-y_1)]^{n-1}\ln[i(y-y_1)]/(n-1)!, \qquad v=n, n=1, 2, 3..., \tag{8}$$

$$A\pi\delta(y-y_1) + A[i(y-y_1)]^{-1}, \qquad v=0, \tag{9}$$

where $\delta(y-y_1)$ denotes the unit impulse function. Also $\arg[i(y-y_1)]$ is $\pi/2$ when $y>y_1$ and $-\pi/2$ $y<y_1$. Therefore we have for $y>y_1$

$$\begin{aligned} i(y-y_1) &= i\,|y-y_1|, \\ \ln[i(y-y_1)] &= \ln|y-y_1| + i\pi/2, \end{aligned} \tag{10}$$

and for $y < y_1$,

$$
\begin{aligned}
i(y-y) &= i^{-1}\,|y-y_1|,\\
\ln[i(y-y_1)] &= \ln|y-y_1| - i\pi/2.
\end{aligned}
\tag{11}
$$

Powers of i are interpreted as powers of $\exp(i\pi/2)$.

4. *Examples*

The following examples show how our results can be used to obtain information regarding the behavior of $F(y)$ near its singularities when $f(u)$ is asymptotically equal to the sum of terms of the form $Au^{-\nu}\exp[iuy_1]$.

(a) $f(u) = (1+u)^{-1/2}$. When $u \to \infty$ the behavior of $f(u)$ is given by the binomial expansion

$$f(u) = u^{-1/2} - \tfrac{1}{2}\,u^{-3/2} + \tfrac{3}{8}\,u^{-5/2} - \cdots, \qquad u > 1. \tag{12}$$

and comparison with the asymptotic series (2) for $f(u)$ gives $y_1 = 0$; $A_1 = 1$, $\nu_1 = \frac{1}{2}$; $A_2 = -\frac{1}{2}$, $\nu_2 = \frac{3}{2}$; $A_3 = \frac{3}{8}$, $\nu_3 = \frac{5}{2}$; ....

$F(y)$ has a singularity at $y = 0$ because $y_1 = 0$. Putting $y_1 = 0$ in (7) and substituting A_1, ν_1 and A_2, ν_2 for A, ν gives the first two terms in the irregular part of $F(y)$

$$
\begin{aligned}
\phi_1(y) &= (1)\,\pi[iy]^{-1/2}/\Gamma(\tfrac{1}{2}) = (\pi/|y|)^{1/2} i^{\mp 1/2},\\
\phi_2(y) &= (-\tfrac{1}{2})\,\pi[iy]^{1/2}/\Gamma(\tfrac{3}{2})(-1) = (\pi|y|)^{1/2} i^{\pm 1/2},
\end{aligned}
\tag{13}
$$

where the upper sign in the exponents of i refers to $y > 0$ and the lower sign to $y < 0$. Setting $M = 2$ in (3) shows that

$$
\begin{aligned}
F(y) &= \int_0^\infty e^{-iuy}(1+u)^{-1/2}\,du\\
&= \phi_1(y) + \phi_2(y) + \psi_2(y)\\
&= (\pi/2)^{1/2}[(1 \mp i)\,|y|^{-1/2} + (1 \pm i)\,|y|^{1/2}] + \psi_2(y),
\end{aligned}
\tag{14}
$$

where we have used $i^{1/2} = \exp(i\pi/4) = (1+i)/2^{1/2}$. When we put $M = 2$ in (4) and note that $\nu_3 = \frac{5}{2}$ we see that $N = 1$. Therefore $\psi_2(y)$ and its first derivative are continuous at $y = 0$.

In order to obtain the actual value of $\psi_2(y)$ at $y = 0$ further investigation is required. Thus, subtracting the leading term in the asymptotic series for $f(u)$ from the integrand in (14) and setting $y = 0$ give

$$\psi_2(0) = \int_0^\infty [(1+u)^{-1/2} - u^{-1/2}]\,du = -2.$$

For this example $F(y)$ can be expressed in terms of Fresnel integrals.

(b) $f(u) = \sin u/u$. Here $F(y)$ has two singularities in contrast to example (a), where there was only one. The "asymptotic expansion" of $f(u)$ consists of two "series,"

$$f(u) = e^{iu}\left(\frac{1}{2iu} + 0 + 0 + \cdots\right) + e^{-iu}\left(\frac{-1}{2iu} + 0 + 0 + \cdots\right) \tag{15}$$

and comparison with (2) shows that one of the y_1's is $+1$ and the other is -1. Therefore $F(y)$ has singularities at $y = +1$ and $y = -1$. At $y = +1$ we have $A_1 = 1/(2i)$, $v_1 = 1$; and (8) with $n = 1$ and $y_1 = 1$ gives

$$\phi_1(y-1) = \frac{i}{2}\ln|y-1| \mp \frac{\pi}{4}, \tag{16}$$

where the upper (–) sign refers to $y > 1$ and the lower (+) sign to $-1 < y < 1$.

Therefore near $y = 1$

$$\begin{aligned} F(y) &= \int_0^\infty e^{-iyu}\sin u\, du/u \\ &= \frac{i}{2}\ln|y-1| \mp \frac{\pi}{4} + [\psi_\infty(y)]_1, \end{aligned} \tag{17}$$

where $[\psi_\infty(y)]_1$ denotes $\psi_M(y)$ for $M = \infty$ and y in the interval $-1 < y < \infty$. The function $[\psi_\infty(y)]_1$ and all of its derivatives are continuous at $y = 1$. From (17) and the known value

$$\begin{aligned} F(y) &= \frac{i}{2}\ln\left|\frac{y-1}{y+1}\right|, & |y| > 1, \\ &= \frac{i}{2}\ln\left|\frac{y-1}{y+1}\right| + \frac{\pi}{2}, & |y| < 1, \end{aligned} \tag{18}$$

it can be shown that

$$[\psi_\infty(y)]_1 = \frac{\pi}{4} - \frac{i}{2}\ln(y+1), \qquad -1 < y < \infty. \tag{19}$$

References

1. D. V. Widder, "The Laplace Transform," pp. 180–183, Princeton Univ. Press, Princeton, N.J., 1946.
2. G. Doetsch, "Handbuch der Laplace-Transformation," Vol. I, pp. 455–460, Vol. II, pp. 97–100, Birkhaüser, Basel, 1950, 1955.
3. R. A. Handelsman and J. S. Lew, Asymptotic expansions of Laplace transforms near the origin, *SIAM J. Math. Anal.* **1** (1970), 118–130.

4. R. A. HANDELSMAN AND J. S. LEW, Asymptotic expansion of a class of integral transforms with algebraically dominated kernels, *J. Math. Anal. Appl.* **35** (1971), 405–433.
5. N. BLEISTEIN, R. A. HANDELSMAN, AND J. S. LEW, Functions whose Fourier transforms decay at infinity; An extension of the Riemann–Lebesgue Lemma, *SIAM J. Math. Anal.* **3** (1972), 485–495.

RECEIVED: March 1, 1978; REVISED: October 31, 1979

ROBERT LUGANNANI AND STEPHEN RICE
Department of Electrical Engineering and Computer Science
University of California, San Diego
La Jolla, California 92093